Eur. Phys. J. D **39**, 93–99 (2006)
DOI: 10.1140/epjd/e2006-00080-8

THE EUROPEAN
PHYSICAL JOURNAL D

The study of soliton fission and fusion in (2+1)-dimensional nonlinear system

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Received 5 September 2005 / Received in final form 13 January 2006
Published online 11 April 2006 –

Abstract. By means of a variable separation approach and an extended homogeneous balance method, a general variable separation excitation of a (2+1)-dimensional nonlinear system is derived. Based on the derived solution with arbitrary functions, we reveal soliton fission and fusion phenomena in the (2+1)-dimensional soliton system.

PACS. 05.45.Yv Solitons – 04.30.Nk Wave propagation and interactions – 02.30.Jr Partial differential equations

1 Introduction

In the study of nonlinear science, soliton theory plays a very important role and has been applied in almost all the natural sciences especially in all the physics branches such as condensed matter physics, field theory, fluid dynamics, plasma physics, nonlinear optics, etc. [1]. Usually, the collisions between solitons of integrable models are regarded to be completely elastic. That is to say, the amplitude, velocity and wave shape of a soliton do not change after nonlinear interaction [2,3]. However, for some special solutions of certain (2+1)-dimensional models in our recent study, the interaction among solitonic excitations like peakons and compactons are not completely elastic since their shapes or amplitudes are changed after their collisions [4]. Furthermore, for some (1+1)-dimensional models, two or more solitons may fuse into one soliton at a special time while for sometimes one soliton may fission into two or more solitons at other special time [5]. We call these two types of phenomena soliton fusion and soliton fission respectively. Actually, the soliton fusion and fission phenomena have been observed in many physical systems such as organic membrane and macromolecular material [6], and physical fields like plasma physics, nuclear physics and hydrodynamics [7]. Recently, Wang et al. [8] discussed two (1+1)-dimensional models, the Burgers equation and the Sharma-Tasso-Olver equation, via the Hirota's direct method, and found the soliton fission and soliton fusion phenomena. Now an interesting and important problem is that are there soliton fission and fusion phenomena in higher dimensions? The main purpose of our present paper is searching for some possible soliton fission and soliton fusion phenomena in three dimensions. As a concrete example, we consider following (2+1)-dimensional asymmetric Nizhnik-Novikov-Veselov (ANNV) equations

$$u_t + u_{xxx} - 3(uv)_x = 0, \tag{1}$$

$$u_x = v_y, \tag{2}$$

which was first derived by Boiti et al. [9] using the weak Lax pair. In fact, equations (1) and (2) can also be obtained from the inner parameter-dependent symmetry constraint of the KP equation [10] and may be considered as a model for an incompressible fluid where u and v are the components of the (dimensionless) velocity [11]. The spectral transformation for this system has been investigated in references [9,12]. This system has also been considered in [13] as a generalization to (2+1)-dimensions of the results from Hirota and Satsuma [14]. The nonclassical symmetries, Painlevé property and similarity solution of the system have been studied by Clarkson and Mansfield [15]. The conditional similarity reductions are studied in [16] and some different types of dromion solutions are given in [17].

2 General variable separation solution to the (2+1)-dimensional ANNV equations

To find some exact explicit soliton solutions for integrable models is one of the most important and significant task. Various effective methods like the inverse scattering transformation (IST), Bäcklund transformation, Darboux transformation, Hirota's direct method, tanh method, sine-cosine method, extended homogeneous balance method, variable separation approach, mapping transformation method and so on [1,18–24].

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In order to derive new solutions with certain arbitrary functions of ANNV equations (1) and (2), we apply the variable separation approach based on extended homogeneous balance method to it. Because the solving processes are similar to that of reference [4], we omit here and only list the final results as follows

$$u = \frac{2\left(a_1 a_2 - a_0 a_3\right) p_x q_y}{\left(a_0 + a_1 p + a_2 q + a_3 pq\right)^2}, \tag{3}$$

$$v = \frac{2\left(a_1 + a_3 q\right)^2 p_x^2}{\left(a_0 + a_1 p + a_2 q + a_3 pq\right)^2} - \frac{2\left(a_1 + a_3 q\right) p_{xx}}{\left(a_0 + a_1 p + a_2 q + a_3 pq\right)} + v_0, \tag{4}$$

which is the same as the results made in reference [25] by multi-linear variable separation approach. In equations (3) and (4), $p \equiv p(x,t)$ is arbitrary functions of $\{x, t\}$, a_0, a_1, a_2, and a_3 are arbitrary constants while v_0 is fixed by

$$v_0 = -\frac{-p_t - p_{xxx} + c_1 a_0 + \left(a_1 c_1 + a_2 c_0 + c_2 a_0\right) p + \left(a_1 c_2 + a_3 c_0\right) p^2}{3p_x}. \tag{5}$$

However, the function $q = q(y,t)$ now is not an arbitrary function and should be determined by the following Riccati equation

$$-q_t + c_0 a_0 + \left(a_1 c_1 + a_2 c_0 - c_2 a_0\right) q - \left(a_2 c_2 - a_3 c_1\right) q^2 = 0, \tag{6}$$

where $\{c_0, c_1, c_2\}$ are arbitrary functions of t. If we select the functions c_0, c_1, and c_2 as

$$c_0 = \frac{1}{a_0 A_1}\left(A_1 A_{2t} - A_2 A_{1t} - A_2^2 A_{3t}\right), \tag{7}$$

$$c_1 = \frac{1}{a_0 A_1\left(a_0 a_3 - a_1 a_2\right)}\Big[a_2^2 A_1 A_{2t} - a_2\left(a_0 + a_2 A_2\right) A_{1t} - \left(a_0 + a_2 A_2\right)^2 A_{3t}\Big], \tag{8}$$

$$c_2 = \frac{1}{a_0 A_1\left(a_0 a_3 - a_1 a_2\right)}\Big[a_2 a_3 A_1 A_{2t} - a_3\left(a_0 + a_2 A_2\right) A_{1t} - \left(a_0 a_1 + a_2 a_3 A_2^2 + 2a_0 a_3 A_2\right) A_{3t}\Big], \tag{9}$$

then the Riccati equation (6) has a solution

$$q = \frac{A_1}{A_3 + F} + A_2, \tag{10}$$

where $A_1 \equiv A_1(t)$, $A_2 \equiv A_2(t)$, $A_3 \equiv A_3(t)$, and $F \equiv F(y)$ are arbitrary functions of the indicated variables.

3 Soliton fission and fusion phenomena in (2+1)-dimensional nonlinear system

As the arbitrariness of characteristic functions $p(x,t)$ and $q(y,t)$ is included in the field (3), u possesses quite rich structures. In reference [25], some types of special stable localized excitations for the ANNV system have been obtained by selecting the arbitrary functions appropriately. Recently, it is reported both theoretically and experimentally that fission and fusion phenomena can happen for (1+1)-dimensional solitons or solitary waves [8], Zheng and Fang study soliton fission and fusion for some (2+1)-dimensional system [26]. However, our nature is colorful and may deduce that the field u would exist some novel properties that have not been revealed. Now we focus our attention on these intriguing fusion and fission phenomena for the field u in (2+1)-dimensions, which may exist in certain situations. For instance, when we select the arbitrary function $p(x,t)$ and $q(y,t)$ to be

$$p = 1 + 2\exp(x - 2t) + \begin{cases} \exp(x+t), & x + t \le 0, \\ -\exp(-x-t) + 2, & x + t > 0, \end{cases} \tag{11a}$$

$$q = 1 + \exp(2y), \tag{11b}$$

and $a_0 = 1$, $a_1 = 0$, $a_2 = 0$, $a_3 = 1$ in equation (3), then we can obtain a new kind of fission solitary wave solution for the field u. Figure 1 shows an evolutional profile of the solitary wave solution for the corresponding field u (3), which depicts a fission phenomenon. From Figure 1, one can clearly see that the one soliton fissions into two solitons. It is interesting to mention that the left travelling soliton along with the positive x-axis and the right travelling soliton along with the negative x-axis, i.e., the pairs of solitons that emerge after the fission, are stable and do not undergo additional fissions as running program for longer periods of time till to $t = 10^3$ (see Figs. 1d and 1e)

Along with the above line, when we consider $p(x,t)$ and $q(y,t)$ to be

$$p = \frac{\exp(5x + 5t) + 0.8\exp(2x + 3t) + 0.5\exp(2x + 4t)}{\left(1 + \exp(2x + 3t)\right)^2}, \tag{12a}$$

$$q = \exp(2y + 3), \tag{12b}$$

and $a_0 = a_1 = a_2 = 0.5$, $a_3 = 1$ for the solution u expressed by equation (3), then we can obtain another new type of fission solitary wave. From Figure 2, we can find that one single soliton fissions into three solitons.

If we select $p(x,t)$ and $q(y,t)$ to be

$$p = \left(1 + \exp(2x + 3t)\right)^2 + \frac{0.8\exp(2x - 3t)}{\left(1 + \exp(2x - 3t)\right)^2}, \tag{13a}$$

$$q = \exp(2y + 3), \tag{13b}$$

and $a_0 = a_1 = a_2 = 0.5$, $a_3 = 1$ for the solution u expressed by equation (3), then we can obtain a new type of fusion solitary wave solution for the field u, which possesses apparently different property compared with Figures 1 and 2. From Figure 3, one can find that two solitons fuse into one soliton finally. The fused single soliton remain stable for subsequent times as we run program for rather long times ($t = 10^3$).

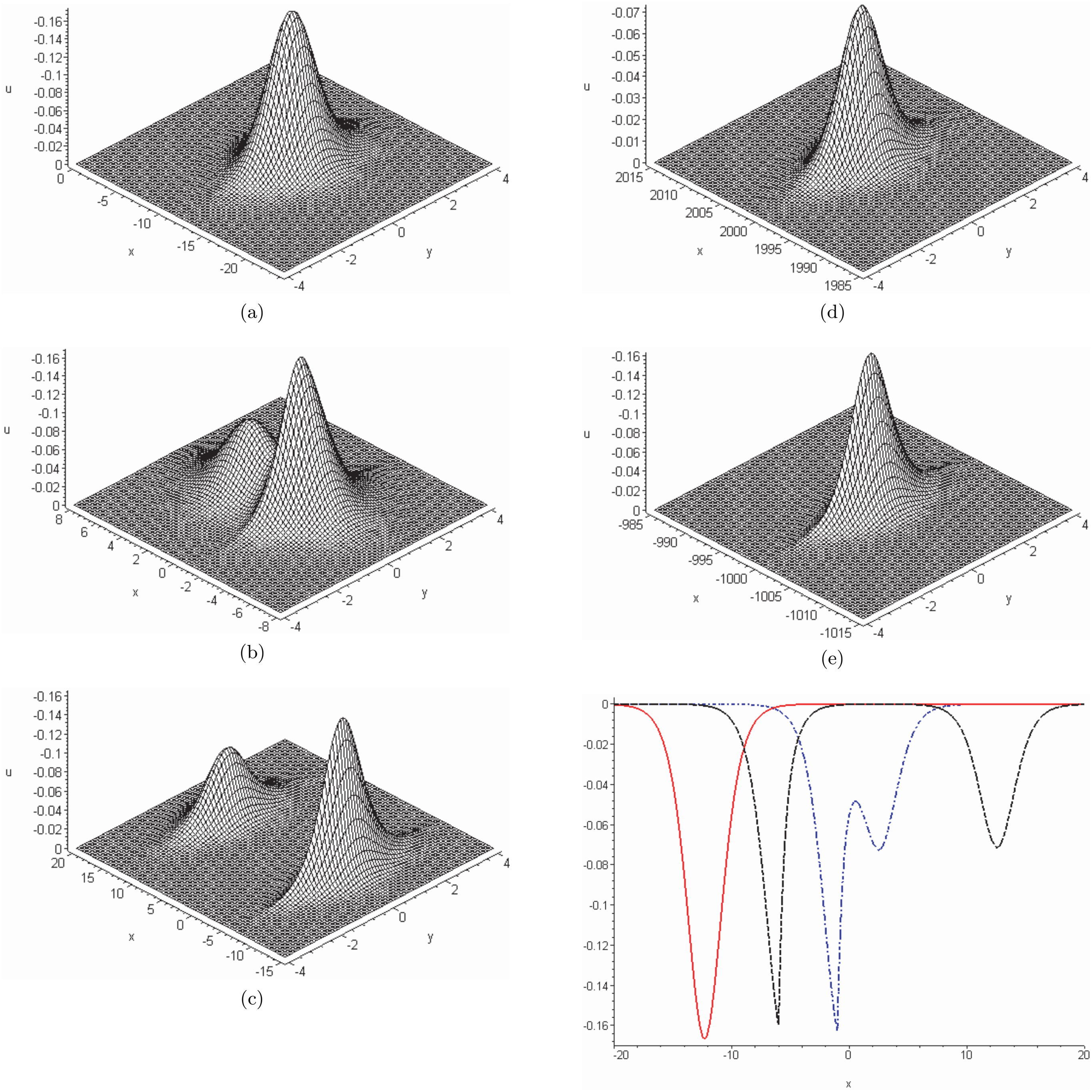


Fig. 1. The evolutional profile of one soliton fission into two solitons for the field u (3) with condition (11) at different times: (a) $t = -6$, (b) $t = 1$, (c) $t = 6$, (d) $t = 1000$, (e) $t = 1000$. (f) A sectional view related to (a–c) at $y = 0$: solid (red) line (a), dotted and dashed (blue) line (b) and dashed (black) line (c).

Along with the above line, when we consider $p(x,t)$ and $q(y,t)$ to be

$$p = 1 + \frac{\exp(5x - 5t) + 0.8\exp(2x - 3t) + \exp(2x - 4t)}{(1+\exp(2x-3t))^2}, \tag{14a}$$

$$q = 1 + \exp(2y), \tag{14b}$$

and $a_0 = 1$, $a_1 = 0$, $a_2 = 0$, $a_3 = 1$ in equation (3), then we can obtain another new type of fusion solitary wave solution for the field u. From Figure 4, one can find that three solitons fuse into one soliton finally. The fused single soliton remain stable for subsequent times as we run program for rather long times ($t = 10^3$).

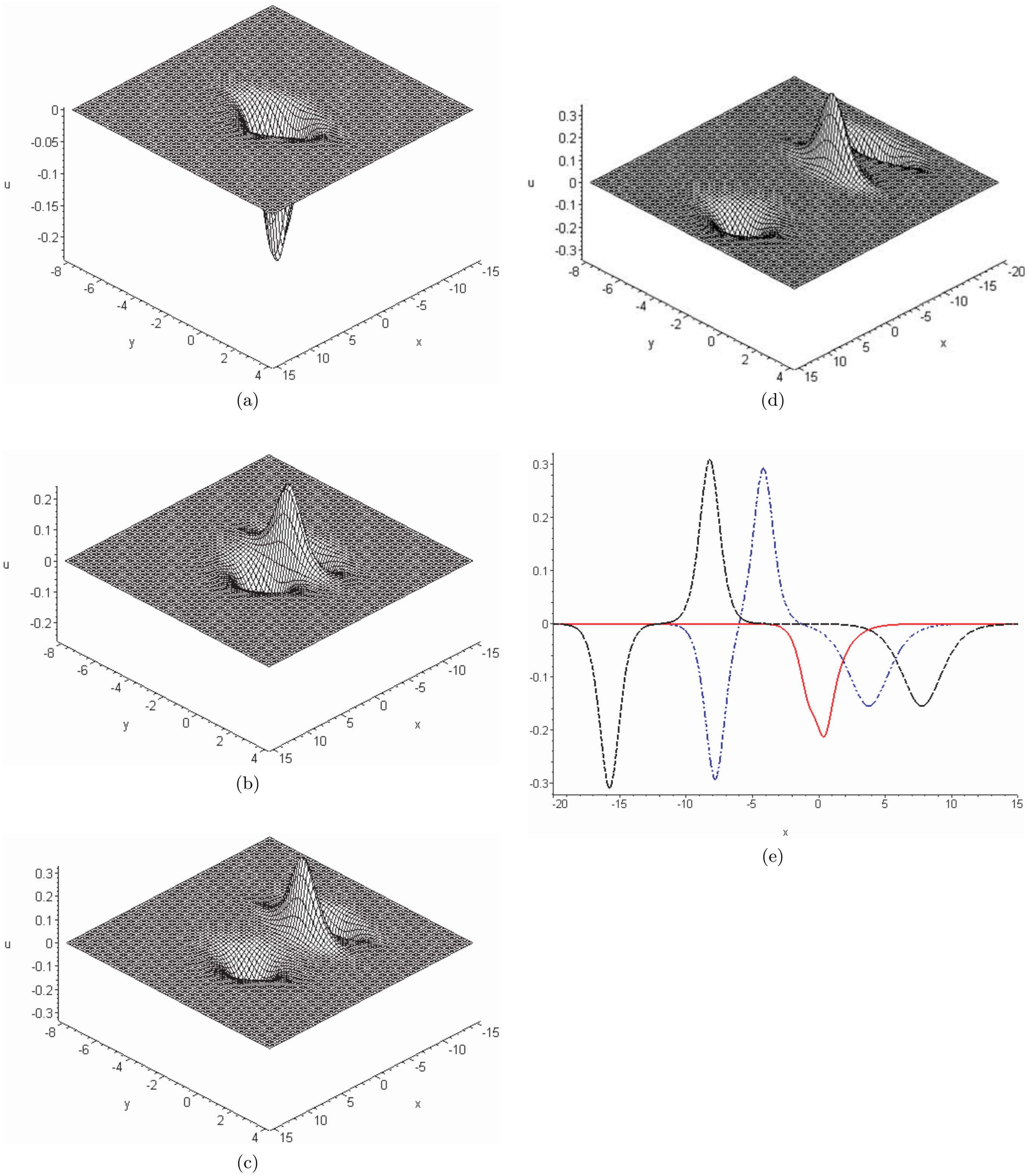


Fig. 2. The evolutional profile of one soliton fission into three solitons for the field u (3) with condition (12) at different times: (a) $t = 0$, (b) $t = 2$, (c) $t = 4$, (d) $t = 8$. (e) A sectional view related to (a), (c) and (d) at $y = -2$: solid (red) line (a), dotted and dashed (blue) line (c) and dashed (black) line (d).

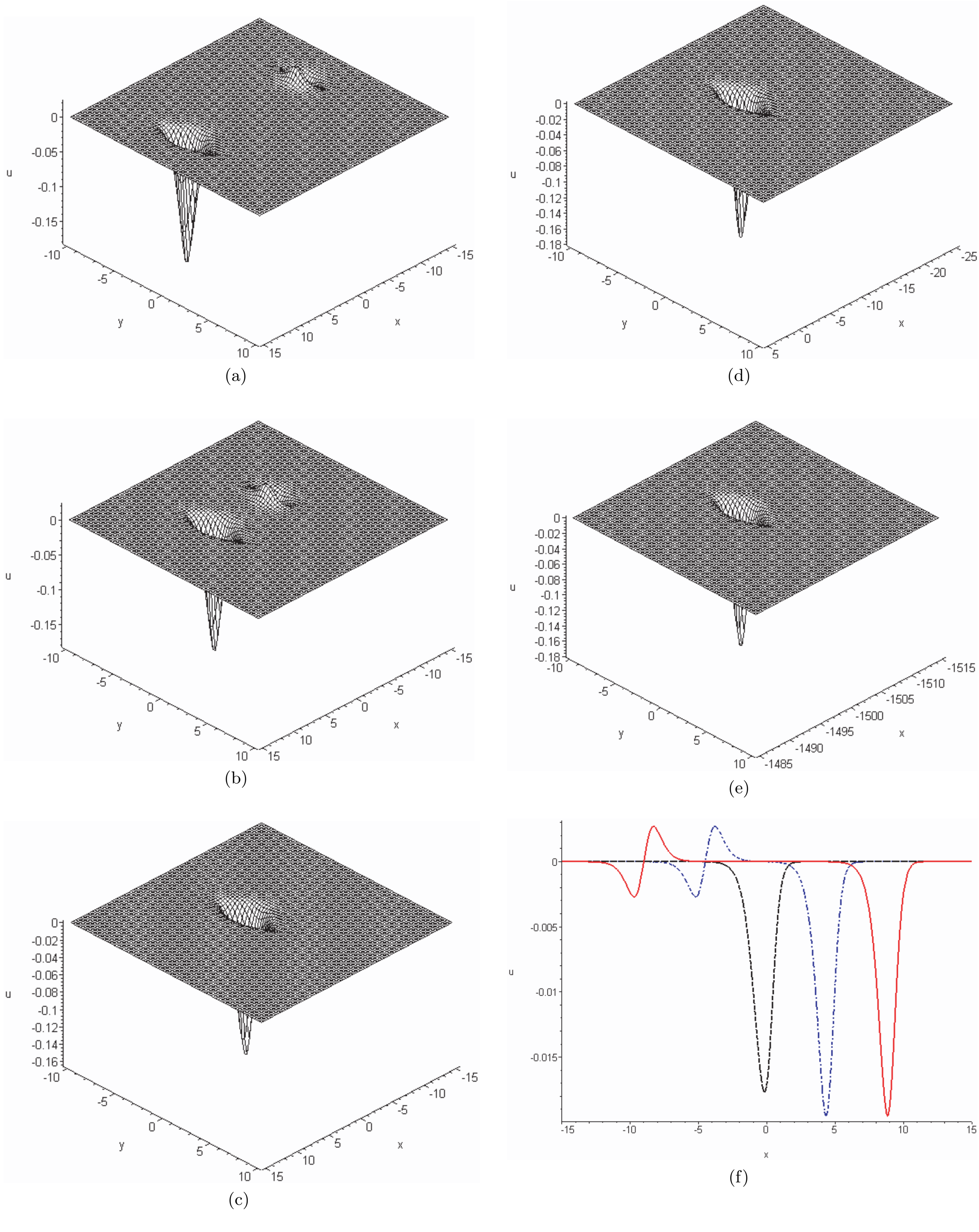
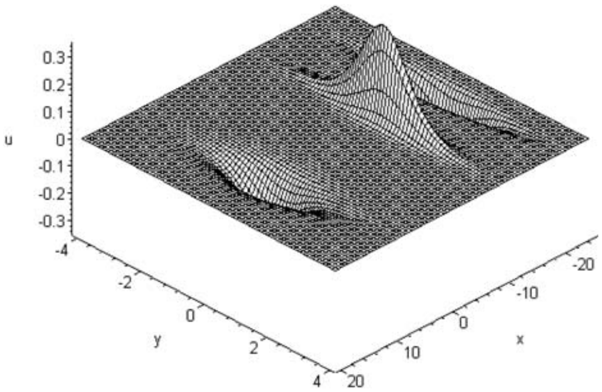
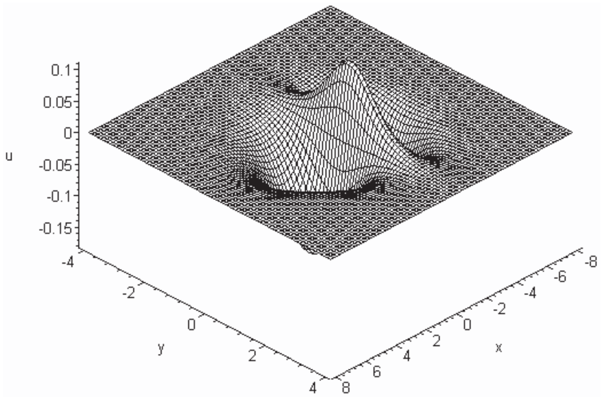


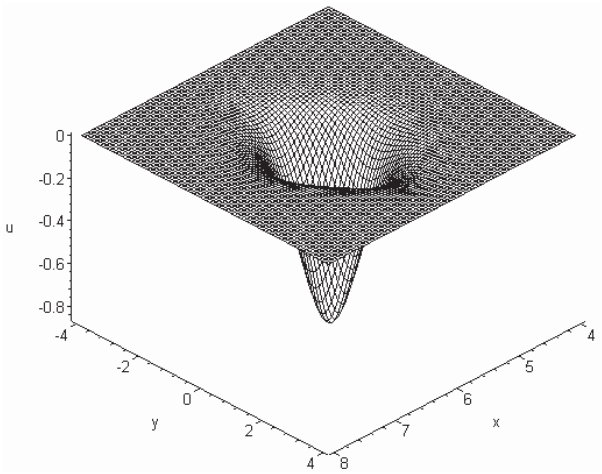
Fig. 3. Two solitons fuse into one soliton evolutional plot of the field u (3) with condition (13) at different times: (a) $t = -6$, (b) $t = -3$, (c) $t = 0$, (d) $t = 6$, (e) $t = 1000$. (f) A sectional view related to (a–c) at $y = 0$: solid (red) line (a), dotted and dashed (blue) line (b) and dashed (black) line (c).



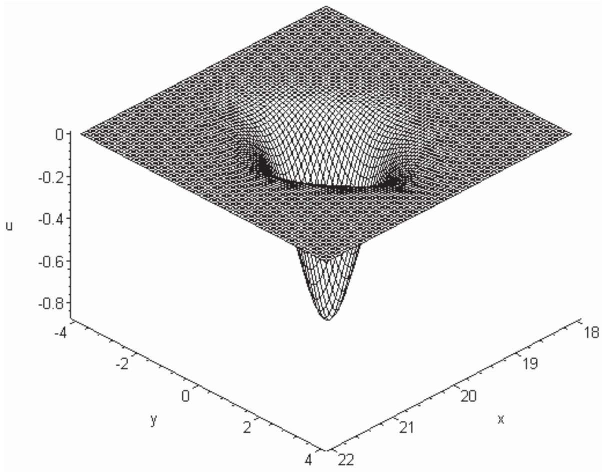
(a)



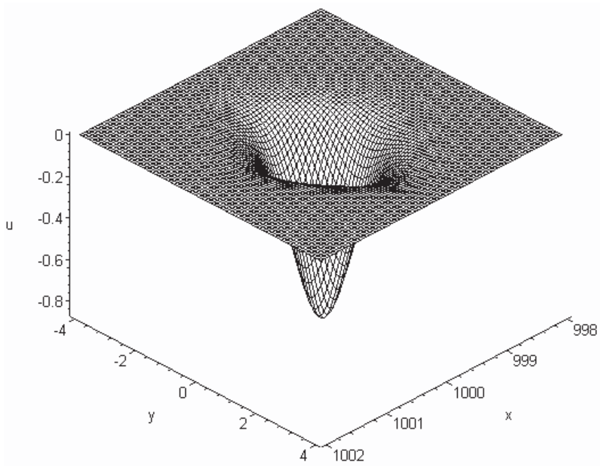
(b)



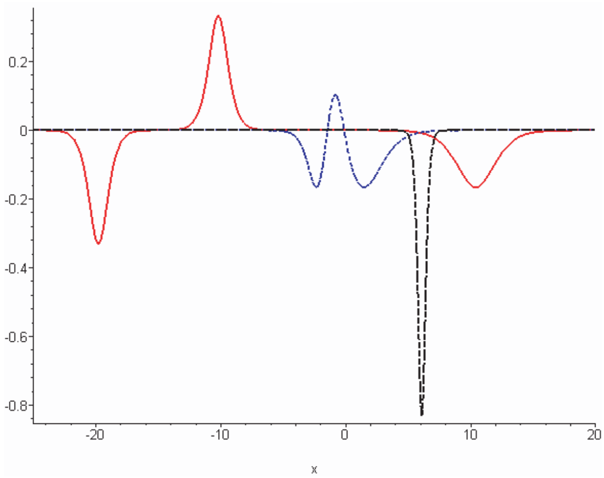
(c)



(d)



(e)



(f)

Fig. 4. Three solitons fuse into one soliton evolutional plot of the field u (3) with condition (14) at different times: (a) $t = -10$, (b) $t = -1$, (c) $t = 6$, (d) $t = 20$, (e) $t = 1000$. (f) A sectional view related to (a–c) at $y = 0$: solid (red) line (a), dotted and dashed (blue) line (b) and dashed (black) line (c).

4 Summary and discussion

In summary, with help of an extended homogeneous balance method and a variable separation approach, the (2+1)-dimensional ANNV equation is solved. Starting from the general variable separation solution with two arbitrary functions, we list some special examples such as soliton fission and fusion solutions for the higher-dimensional ANNV system. Usually, it is considered that the interactions among solitons are completely elastic. However, in some special cases, the soliton interactions may be nonelastic, and for some other situations, the interactions between solitons are completely nonelastic. From the above analysis, we can see that these intriguing phenomena like the soliton fission and fusion may occur in one higher-dimensional soliton system if we choose appropriate initial conditions or boundary conditions. Because of the wide applications of the soliton theory, this paper is only a beginning work. To learn more about the fission and fusion properties and their applications in reality is worth further study.

The authors would like to express their sincere thanks to referees for their many helpful advices and suggestions. The project is supported by the National Natural Science Foundation of China and the Natural Science Foundation of Shandong Province in China.

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